BIBD is not available for all treatments as it has to satisfy the following condition $vr = bk$, $\lambda(v-1) = r(k-1)$, $b \geq v$. Again lattice design is available only for the square number of treatments and cubic number of treatments while youden square design required large number of replication size. So to have an IBD with all possible treatments and a smaller replication size, we required another class of incomplete block design. For such design Bose and Nair (1936) introduce the concept of PBIBD. Which is available for all possible number of treatments and a smaller number of replication size.
$1^{st}$ associate: $2, 3, 4, 6, 8, 10$

$2^{nd}$ associate: $5, 7, 9$

$1^{st}$ associate: $1, 3, 4, 7, 8, 9$

$2^{nd}$ associate: $5, 6, 10$

$1^{st}$ associate: $1, 2, 4, 5, 9, 10$

$2^{nd}$ associate: $6, 7, 8$

$1^{st}$ associate: $1, 2, 3, 5, 6, 7$

$2^{nd}$ associate: $8, 9, 10$

$1^{st}$ associate: $3, 4, 6, 7, 9, 10$

$2^{nd}$ associate: $1, 2, 8$

$1^{st}$ associate: $1, 4, 5, 7, 8, 10$

$2^{nd}$ associate: $2, 3, 9$

$1^{st}$ associate: $2, 4, 5, 6, 8, 9$

$2^{nd}$ associate: $1, 3, 10$

$1^{st}$ associate: $1, 2, 6, 7, 9, 10$

$2^{nd}$ associate: $3, 4, 5$

$1^{st}$ associate: $2, 3, 5, 7, 8, 10$

$2^{nd}$ associate: $3, 4, 5$

$1^{st}$ associate: $1, 3, 5, 6, 8, 9$

$2^{nd}$ associate: $2, 4, 7$

$\therefore n_1 = 6 \quad n_2 = 3$
\[ n_1 + n_2 = 6 + 3 = 9 \]
\[ v - 1 = 10 - 1 = 9 \]
\[ n_1 + n_2 = v - 1 \]

Let there are \( v \) treatments denoted as 1, 2, ..., \( v \). These treatments follow association scheme, if it satisfies the following:

(i) For a given treatment \( \theta \), there are \( i^{th} \) associate treatments.

(ii) For a given treatment \( \theta \), \( i^{th} \) associate treatments occur \( n_i \) times.

(iii) Pair of treatments occurs together \( \lambda_i \) times. For any two treatment, say, \( \theta \) and \( \phi \), number of common treatment between \( i^{th} \) associate of \( \theta \) and \( j^{th} \) associate of \( \phi \) is constant and is denoted by \( p_{ij} \) matrix which is given by
\[ P_{ij}^k = \begin{pmatrix} P_{11}^k & \cdots & P_{1m}^k \\ \vdots & \ddots & \vdots \\ P_{m1}^k & \cdots & P_{mm}^k \end{pmatrix} \]

where \( P_{ij}^1 \) and \( P_{ij}^2 \) \( \ldots \), \( (k = 1, 2, \ldots, m) \) are called association matrix of \( 1^{\text{st}} \) and \( 2^{\text{nd}} \) \( \ldots \), \( m^{\text{th}} \) associate classes and the whole scheme is called association scheme.

Suppose treatments 1 and 2 are \( 1^{\text{st}} \) associate then .

\[ \therefore p'_{11} = 3, \quad p'_{12} = 2, \quad p'_{21} = 2, \quad p'_{22} = 1. \]

\[ \begin{array}{ccc}
1 & \rightarrow & 2, 3, 4, 6, 8, 10 \\
& & 5, 7, 9 \\
& \rightarrow & 1, 3, 4, 7, 8, 9 \\
2 & & 5, 6, 10.
\end{array} \]

\[ \therefore p'_{11} = \text{number of common treatment between 1(1), and 2(1) } = \{3, 4, 8\} = 3 \]

\[ \therefore p'_{22} = \text{number of common treatment between 1(2), and 2(2) } = \{5\} = 1 \]

\[ p_{12}^{(1)} = p_{21}^{(1)} = \text{number of common treatment between 1(2) and 2(1) } = \{7, 9\} = 2. \]
1(1) and 2(2) common = \{6,10\} = 2

\[
\therefore p'_{ij} = \begin{pmatrix} p'_{11} & p'_{12} \\ p'_{21} & p'_{22} \end{pmatrix} \therefore p'_{12} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}
\]

Now consider treatments 1 and 5 are 2\textsuperscript{nd} associate.

(1) \[1 \rightarrow (2,3,4,6,8,10) \text{ A (5,7,9) C} \]
\[5 \rightarrow (4,6,7,3,9,10) \text{ B (1,2,8) D} \]

\[
\therefore 2\textsuperscript{nd} \text{ associate}
\]
\[
p^2_{11} = A \cap B = \{6,10,3,10\} = 4
\]
\[
p^2_{12} = A \cap D = \{2,8\} = 2
\]
\[
p^2_{21} = B \cap C = \{7,9\} = 2 \therefore p^2_{12} = p^2_{21}
\]
\[
p^2_{22} = C \cap D = \{\phi\} = 0
\]

\[
\therefore p^2_{ij} = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}
\]
This shows that associate matrix are symmetric matrix.

Now, \[ p_{11}^1 + p_{12}^1 + p_{21}^1 + p_{22}^1 = 3 + 2 + 2 + 1 = 8 \]

\[ \therefore \sum_{i=j=1}^{2} p_{ij}^1 = v - 2 \]

Also, \[ \therefore \sum_{i=j=1}^{2} p_{ij}^2 = v - 2 \]

\[ \therefore \sum_{i=1}^{m} n_i = v - 1 \]
Definition: Partially Balanced Incomplete Block Design (PBIBD)
An incomplete block design is said to be PBIB Design if $v$ treatments are arranged in $b$ blocks, each block contains $k$ treatments ($k<v$), each treatment occurs in $r$ blocks and a pair of treatments occur together in $\lambda_i$ blocks ($i = 1, 2, \ldots, m$), provided it follows the following association schemes.

- In association scheme, there are $i^{th}$ classes.
- For a given treatment, say, $\theta$, $n_i$ treatments occur in $i^{th}$ associate class.
- For any given treatment, say, $\theta$, the remaining treatments, if they are $i^{th}$ associate class, occur together in $\lambda_i$ blocks.
- For any two treatments, say, $\theta$ and $\phi$, number of common treatment between $i^{th}$ associate of $\theta$ and $j^{th}$ associate of $\phi$ is constant and is denoted by $p_{ij}$ matrix.
Relation between BIBD and PBIBD.
If $\lambda_1 = \lambda_2 = \ldots = \lambda_i = \lambda$ then PBIBD becomes BIBD.

Difference between BIBD and PBIBD.

<table>
<thead>
<tr>
<th>BIBD</th>
<th>PBIBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pair of treatments occur $\lambda$ times</td>
<td>Pair of treatments occur $\lambda_i$ times</td>
</tr>
<tr>
<td>$b \geq v$</td>
<td>dose not hold</td>
</tr>
<tr>
<td>dose not hold association schemes</td>
<td>Hold association schemes.</td>
</tr>
</tbody>
</table>

Parameters of PBIBD.
On the basis of association schemes, PBIBD has two types of parameters.

(1) Primary parameters: $v, b, r, k, \lambda_i$ \quad (i = 1, 2, \ldots, m)
(2) Secondary parameters: $n_i, P_{ij}^k$ \quad i \neq j = (1, 2, \ldots, v)
\quad k = 1, 2, \ldots, m

Parametric Relation.

(1) $vr = bk$  
(2) $\sum_{i=1}^{n} n_i = v - 1$  
(3) $\sum_{i=1}^{m} n_i \lambda_i = r(k - 1)$
prove that \( \sum n_i = v - 1 \)

For any PBIB Design with \( m^{th} \) association schemes we know that

\[
\sum_{i=0}^{m} B_i = E_{vv}
\]

now   post multiply both said by \( E_{v1} \)

\[
\sum_{i=0}^{m} B_i E_{v1} = E_{vv} E_{v1}
\]

Where \( B_i \) is a matrix of order \( v \times v \) and is called association matrix since every associate, i.e., \( i^{th} \) associate has \( n_i \) treatment.

\[
\therefore \sum_{i=0}^{m} n_i E_{v1} = v E_{v1} \Rightarrow \sum_{i=0}^{m} n_i = v
\]

\[\Rightarrow n_o + \sum_{i=1}^{m} n_i = v \quad \text{where } n_o = 1\]

\[\Rightarrow \sum_{i=1}^{m} n_i = v - 1\]
Prove that: \( \sum_{i=1}^{m} n_i \lambda_i = r(k - 1) \)

For any PBIB Design with \( m^{th} \) association scheme,

we know that: \( \sum_{i=0}^{m} B_i = E_{vv} \) (1) and

\[ NN' = \sum_{i=0} \lambda_i B_i \] (2)

Where \( \lambda_i \) is number of times a pair of treatments occur together in \( i^{th} \) associate class. Now post multiply by \( E_{v1} \) of both said of (2).
\[ \therefore NN' E_{v1} = \sum_{i=0}^{m} \lambda_i B_i E_{v1} \quad \therefore N(N' E_{v1}) = \left( \sum_{i=0}^{m} \lambda_i B_i \right) E_{v1} \]

\[ \therefore NkE_{b1} = \sum_{i=0}^{m} \lambda_i n_i E_{v1} \quad \text{or, } \quad kNE_{b1} = \sum_{i=0}^{m} n_i \lambda_i E_{v1} \]

\[ \therefore krE_{v1} = \sum_{i=0}^{m} n_i \lambda_i E_{v1} \Rightarrow kr = \sum_{i=0}^{m} n_i \lambda_i \]

\[ \Rightarrow kr = n_0 \lambda_0 + \sum_{i=1}^{m} n_i \lambda_i \]

\[ \Rightarrow \sum_{i=1}^{m} n_i \lambda_i = kr - n_0 \lambda_0 = kr - r = r(k - 1) \]

\[ \Rightarrow \sum_{i=1}^{m} n_i \lambda_i = r(k - 1) \]
Example :
\( v = b = 4 \), \( r = k = 2 \), \( \lambda = 1 \).

\[ \begin{align*}
1^{st} \text{ associate} & & 2^{nd} \text{ associate} \\
1 & 2 & 1 \rightarrow 2,4 & 3 \\
2 & 3 & 2 \rightarrow 1,3 & 4 \\
3 & 4 & 3 \rightarrow 2,4 & 1 \\
4 & 1 & 4 \rightarrow 1,3 & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
& & & \\
B_0 & B_1 & B_2 \\
\end{align*} \]

Here, \( n_1 = 2 \) \( \therefore \lambda_1 = 1 \), \( n_2 = 1 \) \( \lambda_2 = 0 \).
In \( B_0, B_1 & B_2 \) if the treatment is present, write 1, otherwise 0.
\[
B_0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}_{4 \times 4}
\]
\[
B_1 = \begin{bmatrix}
2 & 4 \\
1 & 3 \\
2 & 4 \\
1 & 3
\end{bmatrix}
\]
\[
= \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

second associate
\[
B_2 = \begin{pmatrix}
3 \\
4 \\
1 \\
2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}_{4 \times 4}
\]
\[
\therefore B_0 = I_4
\]

Now \( B_1 + B_2 + B_3 = \)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\therefore \sum_{i=0}^{2} B_i = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}_{4 \times 4} = E_{44} = E_{\nu \nu}
\]

\[
\therefore \sum_{i=0}^{m} B_i = E_{\nu \nu}
\]

Now we want to prove, \( NN' = \lambda_0 B_0 + \sum_{i=1}^{m} \lambda_i B_i \)
Now $\lambda_1 B_1 + \lambda_2 B_2 = $

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
+ 0
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

\[=\]

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
$$

$\therefore \sum_{i=1}^{2} \lambda_i B_i = 

\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
$

$\lambda_0 B_0 = rB_0 = 2B_0 = 

\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}$
now, $\lambda_0 B_0 + \sum_{i=1}^{2} \lambda_i B_i$

$$\begin{bmatrix}
2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2 \\
\end{bmatrix}$$

(1)

Here, Incident matrix $N =
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 0 & 0 & 1 \\
2 & 1 & 1 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 \\
\end{array}$
\[ N' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \therefore NN' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \]

from (1) and (2)
\[ NN' = \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 \quad \text{here, } \lambda_0 = r \quad \text{always} \]
\[ = rB_0 + \sum_{i=1}^{m} \lambda_i B_i \]

Classification of PBIB design:
Bose and Simamamoto (1952) classified PBIB Design of two associate classes in to following types on the basis of its association schemes.
• Simple PBIB Design with $\lambda_1 = 0$ or $\lambda_2 = 0$.
• Group divisible design.
• Rectangular type PBIB Design.
• Latin square type PBIB Design.
• Cycle PBIB Design.

Again Bose introduced another type of PBIB Design with two associate classes and named it as Partial geometry type PBIBD. The remaining PBIB Design which do not fall under these 6 categories on the basis of their association schemes & other parameters are called PBIB Design of miscellaneous type.

Simple PBIBD Design:
A PBIB design with two associate classes is said to be simple PBIB Design, if either (i) $\lambda_1 \neq 0, \lambda_2 = 0$ or (ii) $\lambda_1 = 0, \lambda_2 \neq 0$.

Group Divisible Design. (GDD):
GD Design is simplest class of PBIB Design. A PBIB Design is called GD Design if $v = mn$ treatments are grouped into $m$ groups each of $n$ treatments such that the treatment belonging
to the same group is called $1^{st}$ associate treatment and treatment belonging to different group are called $2^{nd}$ associate treatments.

The following are the parameters of GD Design.

$$v, b, r, k, \lambda_1, \lambda_2, \quad p_{jk}, m, n$$

Example-3

$$\begin{align*}
\text{Here row wise all pair is called } 1^{st} \text{ associate but if we take (1,4) which is different group so it is } 2^{nd} \text{ associate.}
\end{align*}$$

Here, $m = 3$ (3-rows), $n = 2$ (each row has 2 treatments)

$v = 6 = 3 \times 2 = 6$.

So $v = mn$, $\therefore$ this is GD Design

Parametric relation:

$$n = n_1 + 1$$

$$m = (n_2/n) + 1$$

$$p_{jk}^1 = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix}, \quad p_{jk}^2 = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix},$$
Bose and Connor (1952) characterized group divisible design into three categories on the basis of characteristic roots of $NN'$ matrix of GD Design.

1. Singular group divisible design (SGD Design).
2. Semi regular group divisible design (SRGD Design).
3. Regular group divisible design (RGD Design).

Singular group divisible design (SRGD)

A GD design is called singular group divisible if it satisfies the following characteristic roots of $NN'$ matrix.

\[(i) \quad r-\lambda_1 = 0 \text{ and } rk - v\lambda_2 > 0,\]
on otherwise, non-singular group divisible designs.

Non-singular group divisible designs are either semi regular group divisible or regular group divisible designs.
Semi Regular group divisible design (SRGD):
A GD design is said to be SRGD design if the characteristic root of \( NN' \) matrix satisfy the following:
\[
(i) \ r - \lambda_1 \neq 0 \\
(ii) \ rk - v\lambda_2 = 0.
\]

Regular group divisible design (RGD design):
A GDD is said to be RGD Design if the characteristics root of \( NN' \) matrix satisfy the following condition.
\[
(i) \ r - \lambda_1 > 0 \\
(ii) \ rk - v\lambda_2 > 0.
\]

Example: Identify the design. Write its parameter, obtain C-matrix, \( NN' \) matrix, association matrix and show that.

\[
NN' = \sum_{i=0}^{2} B_i \lambda_i \\
\sum_{i=0}^{2} B_i = E_{vv}
\]
1 2 4 → This is an Incomplete Block Design
2 3 5
3 4 6 → \( v = b = 6 \), \( r = k = 3 \)
4 5 1 
This is a symmetrical IBD
5 6 2
6 1 3 → \( \lambda_1 = 1, \lambda_2 = 2, n_1 = 4, n_2 = 1 \)

1 \( \boxed{2, 3, 5, 6, \ n_1} \)

4 \( \boxed{n_2} \)

\[ p_{ij}^1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \therefore p_{ij}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \]

This is a PBIBD of two associate classes

\[ n = n_1 + 1 \quad \therefore n = 5, \]

\[ m = \frac{n_2}{n} + 1 \quad m = \frac{1}{5} + 1 = \frac{6}{5} \]

\[ v = nm = 5 \cdot \frac{6}{5} = 6 \]

this is not a group divisible design.
\[ r - \lambda_1 = 1 = 2 > 0. \]
\[ r_k - v\lambda_2 = 9 \]

**Incidence matrix** \( N \)

\[
\begin{array}{ccccccc}
\hline
T & B & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
2 & 1 & 1 & 0 & 0 & 1 & 0 \\
3 & 0 & 1 & 1 & 0 & 0 & 1 \\
4 & 1 & 0 & 1 & 1 & 0 & 0 \\
5 & 0 & 1 & 0 & 1 & 1 & 0 \\
6 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]
\[ N' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \]

\[ NK^{-1} = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix}_{6 \times 6} \]

C-matrix \[ C = rI_v - NK^{-1} N' = rI_v - \frac{NN'}{k} \]

\[ C = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix} \]
\[
C = \begin{bmatrix}
2 & -1/3 & -1/3 & -2/3 & -1/3 & -1/3 \\
-1/3 & 2 & -1/3 & -1/3 & -2/3 & -1/3 \\
-1/3 & -1/3 & 2 & -1/3 & -1/3 & -2/3 \\
-2/3 & -1/3 & -1/3 & 2 & -1/3 & -1/3 \\
-1/3 & -2/3 & -1/3 & -1/3 & 2 & -1/3 \\
-1/3 & -1/3 & -2/3 & -1/3 & -1/3 & 2 \\
\end{bmatrix}
\]

\[
\lambda_1 B_1 = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
\[ \lambda_2 B_2 = 2 \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \lambda_0 B_0 = 3B_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \]
now, $\lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix} = NN'$

$\therefore \sum_{i=0}^{2} \lambda_i B_i = NN'$

now, $B_0 + B_1 + B_2 = \sum_{i=0}^{2} B_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = E_{vv}$
\[ \sum_{i=0}^{2} B_i = E_{vv} \]

Further, the \( C \)-matrix can be written as:

\[
3C = \begin{bmatrix}
6 & -1 & -1 & -2 & -1 & -1 \\
-1 & 6 & -1 & -1 & -2 & -1 \\
-1 & -1 & 6 & -1 & -1 & -2 \\
-2 & -1 & -1 & 6 & -1 & -1 \\
-1 & -2 & -1 & -1 & 6 & -1 \\
-1 & -1 & -2 & -1 & -1 & 6 \\
\end{bmatrix}
\]

Now
\[
\begin{bmatrix}
6 & -1 & -1 \\
-1 & 6 & -1 \\
-1 & -1 & 6 \\
\end{bmatrix}
= \begin{bmatrix}
6 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 6 \\
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} - \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[ = 7I_3 - E_{33} \]
similarly

\[
\begin{pmatrix}
-2 & -1 & -1 \\
-1 & -2 & -1 \\
-1 & -1 & -2
\end{pmatrix}
= -2I_3 + I_3 - E_{33}
\]

\[
\therefore C = \left[ \begin{array}{cc}
7I_3 - E_{33} & -I_3 - E_{33} \\
-I_3 - E_{33} & 7I_3 - E_{33}
\end{array} \right] / 3
\]

1. Eigenvalue of C-matrix = 7/3 with multiplicity (3-1) = 2
2. Eigenvalue of C-matrix = 7/3 with m = 2
   \[\therefore Q_1 = Q_2 = Q_3 = Q_4 = 7/3.\]

Other eigenvalue will be obtained by solving this matrix.

If we consider \(n_1 = 1\) and \(n_2 = 4\) then \(n = 2\) and \(m = 3\) so it becomes group divisible designs. Again \(r - \lambda_1 = 3 - 2 > 0\) and \(rk - v \lambda_2 > 0\), so design is regular group divisible design.
A PBIB design with two associate classes is said to be Triangular, if the number of treatments $v = n(n-1)/2$ and the association scheme is arranged in $n$ rows and $n$ columns such that:

(i) The position in the principle diagonal of the scheme are left blank.

(ii) The $n(n-1)/2$ positions above the principal diagonal are filled by the treatment numbers 1, 2, ..., $n(n-1)/2$.

(iii) The $n(n-1)/2$ positions below the diagonal are so filled that the array is symmetrical about the principal diagonal, and

(iv) For any treatment $i$ the first associates are exactly those treatments which lie in the same row as $i$.

\[ n_1 = 2n-4; \quad n_2 = (n-2)(n-3)/2 \]

\[ p_{ij}^1 = \begin{bmatrix} (n-2) & (n-3) \\ (n-3) & (n-3)(n-4)/2 \end{bmatrix} \]

\[ p_{ij}^2 = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix} \]
Triangular type PBIB design
A PBIBD design with two associate classes is said to be triangular, if the number of elements \( v = n(n-1)/2 \) and the association scheme is an arrange of \( n \) rows and \( n \) columns such that:

(i) The position in the principle diagonal of the scheme are left blank.

(ii) The \( n(n-1)/2 \) positions above the principal diagonal are filled by the treatment numbers 1,2,…,\( n(n-1)/2 \).

(iii) The \( n(n-1)/2 \) position below the diagonal are so filled that the array is symmetrical about the principal diagonal.

(iv) For any treatment \( i \) the first associates are exactly those treatment which lie in the same row as \( i \).

\[
\begin{align*}
  n_1 &= 2n-4; & n_2 &= (n-2) (n-3)/2 \\
  p_{ij}^1 &= \begin{bmatrix} (n - 2) & (n - 3) \\ (n - 3) & (n - 3)(n - 4)/2 \end{bmatrix} \\
  p_{ij}^2 &= \begin{bmatrix} 4 & 2n - 8 \\ 2n - 8 & (n - 4)(n - 5)/2 \end{bmatrix}
\end{align*}
\]
LATIN SQUARE TYPE OF PBIBD :-
Let a square array of n rows and n columns be formed with 
\( n^2 \) treatments, numbering from 1 to \( n^2 \) so that two treatments are first associate if they occur in the same row or the same column of the array and the second associates otherwise.
A design with the above array as association scheme is said to belong to the subtype \( L_2 \).

Subtype \( L_3 \): If one can form a square array of \( n^2 \) treatments numbers from 1 to \( n^2 \) and to impose a latin square with n letters on this array, so that any two treatments are first associate if they occur in the same row or column of the array or correspondence to the same letter of the latin square and are second associates otherwise.
In this design the secondary parameters are 

\[
\therefore n_1 = L(n-1), \quad n_2 = (n-1)(n-L+1)
\]
\[ p_{ij}^1 = L^2 - 3L + n \]
\[ (L - 1)(n - L + 1) \]
\[ (L - 1)(n - L + 1) \]
\[ (n - L)(n - L + 1) \]

\[ p_{ij}^2 = L(L - 1) \]
\[ L(n - L) \]
\[ L(n - L) \]
\[ (n - L)^2 + (L - 2) \]

where \( L = 2 \) and \( L = 3 \) for sub type \( L_2 \) and \( L_3 \) respectively.

**Cyclic PBIB Design**

A non group divisible PBIB Design is called cyclic PBIBD if the set of first associates of the treatment numbered \( i \) is obtained by adding \( i - 1 \) to the numbers in the set of first associates of the treatment numbered \( 1 \) and subtracting \( v \) whenever the sum exceeds \( v \).
Example: Find out eigen value of example 1 write its parameters The given design is

\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
3 & 4 \\
4 & 1
\end{bmatrix}
\]

\(n_1 = 1, \lambda_1 = 0, n_2 = 2, \lambda_2 = 1\)

\[
\begin{bmatrix}
0 & 0 \\
0 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\(v = b = 4, \quad r = k = 2.\)

\(n = n_1 + 1 = 1 + 1 = 2\)

\(m = \frac{n_2}{n} + 1 = \frac{2}{2} + 1 = 2\)

\(v = mn = 2 \times 2 = 4\) so the design belong to GD design.

\(r - \lambda_1 = 2 - 0 = 2 > 0\)

\(rk - v\lambda_2 = 2(2) - 4(1) = 0\)

This design is semi regular group divisible design.
We have already proved that,
\[ NN' = \sum_{i=0}^{2} B_i \lambda_i \]

Now C-matrix is given by
\[ C = \text{diag } (r_1 \ldots r_v) - NK^{-1}N' \]
\[ rI_v - \frac{NN'}{k} \quad \{ r_1 = r_2 = r_3 = \ldots = r_v = r \text{ in a PBIB Design} \} \]

\[
2 I_4 - \frac{1}{2} \begin{bmatrix}
2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1/2 & 0 & -1/2 \\
-1/2 & 1 & -1/2 & 0 \\
0 & -1/2 & 1 & -1/2 \\
-1/2 & 0 & -1/2 & 1
\end{bmatrix}
\]
\[
2C = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\]

\[
= \begin{bmatrix} 3I_2 - E_{22} & I_2 - E_{22} \\
I_2 - E_{22} & 3I_2 - E_{22} \end{bmatrix}
\]

\[
\therefore \theta_1 = \frac{3}{2} + \frac{1}{2} = 2, \quad \theta_2 = \frac{3}{2} - \frac{1}{2} = 1
\]

Example:

\[
\begin{bmatrix}
1 & 4 & 2 & 5 \\
2 & 5 & 3 & 6 \\
3 & 6 & 1 & 4
\end{bmatrix}
\]

\[
\begin{align*}
1 & \rightarrow 4 & 2 & 3 & 5 & 6 \\
2 & \rightarrow 5 & 1 & 3 & 4 & 6 \\
3 & \rightarrow 6 & 1 & 2 & 4 & 5 \\
4 & \rightarrow 1 & 2 & 3 & 5 & 6 \\
5 & \rightarrow 2 & 1 & 3 & 4 & 6 \\
6 & \rightarrow 3 & 1 & 2 & 4 & 5
\end{align*}
\]
here, \( v = 6, b = 3, r = 2, k = 4, \lambda_1 = 2 \).
\( n_2 = 4, n_1 = 1, n = n_1 + 1 = 2 \)

\[
m = \frac{n_2}{n} + 1 = \frac{4}{2} + 1 = 3.
\]

\( mn = 2(3) = 6. \)

\( v = 6 = mn \) and \( r - \lambda_1 = 2 - 2 = 0 \)

\[ \therefore \text{This is a singular group divisible design.} \]

**Incidence Matrix** \( N \) can be written as

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}_{3 \times 6}
\]
\[
\begin{bmatrix}
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
\end{bmatrix}
= \begin{bmatrix}
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
\end{bmatrix}
= \begin{bmatrix}
n N' \end{bmatrix}
\]

\[
\therefore \quad \begin{bmatrix} N N' \end{bmatrix} = rB_0 + \lambda_1B_1 + \lambda_2B_2 \\
\]

now, \( C = rI_v - \frac{NN'}{k} = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix} - \frac{1}{4} \begin{bmatrix}
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
\end{bmatrix} \]
\[
\begin{pmatrix}
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2
\end{pmatrix}
\]

\[
NN' = \begin{pmatrix}
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2
\end{pmatrix}_{6\times6}
\]

now, \( rB_0 + \lambda_1 B_1 + \lambda_2 B_2 \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
+ 2
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
+ 1
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}
\]
\[
\begin{bmatrix}
\frac{6}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{2}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{6}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{2}{4} & -\frac{1}{4} \\
-\frac{1}{4} & -\frac{1}{4} & \frac{6}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{2}{4} \\
-\frac{2}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{6}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & -\frac{2}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{6}{4} & -\frac{1}{4} \\
-\frac{1}{4} & -\frac{1}{4} & -\frac{2}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{6}{4}
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & -1 & -1 & -2 & -1 & -1 \\
-1 & 6 & -1 & -1 & -2 & -1 \\
-1 & -1 & 6 & -1 & -1 & -2 \\
-2 & -1 & -1 & 6 & -1 & -1 \\
-1 & -2 & -1 & -1 & 6 & -1 \\
-1 & -1 & -2 & -1 & -1 & 6
\end{bmatrix}
\]

\[
\therefore 4C = \begin{bmatrix}
7I_3 - E_{33} & -I_3 - E_{33} \\
-I_3 - E_{33} & 7I_3 - E_{33}
\end{bmatrix}
\]
Here $\theta_1 = 8/4 = 2$ and $\theta_2 = (8-1)/4 = 7/4$

Intrablock Analysis of a PBIBD:

Let $S_i (t_\alpha) = \text{sum of those treatments which are } i^{\text{th}}$ associate of treatment $t_\alpha$.

Let $S_i (t_\alpha) = \sum_{u=1}^{v} b_{\alpha u}^i t_u = \alpha^{\text{th}}$ element of $B_i$.

and $S_j S_i (t_\alpha) = \text{sum of those treatments which are } j^{\text{th}}$ associate of $i^{\text{th}}$ associates of treatment $t_\alpha$.

Lemma: Show that that for $m$-associate classes PBIBD, $S_j S_i (t_\alpha) = \sum_{u=0}^{m} b_{ji}^k S_u (t_\alpha)$.
Proof: \( B_j B_i \tau = (B_j B_i) \tau = \sum_{u=0}^{m} (p_{ji}^u B_u) \tau \) (if we multiply two associate treatments \( i^{th} \) and \( j^{th} \) then product will give \( \sum p_{ji} \) of the \( u \) associates \( u = 1( ) m \).)

\[ = \sum_{u=0}^{m} p_{ji}^u (B_u \tau) \]

hence \( \alpha^{th} \) element of \( B_j B_i \tau \) is \( = \sum_{u=0}^{m} p_{ji}^u S_u (t_\alpha) \) \hspace{1cm} (1)

also \( B_j B_i \tau = B_j (B_i \tau) \)

\[ = B_j \{ S_i (t_1) S_i (t_2) ... S_i (t_\alpha) ... S_i (t_v) \} \]

hence \( \alpha^{th} \) element of \( B_j B_i \tau \) is \( \sum_{w=1}^{v} b_{\alpha w} S_i (t_w) \)

\[ = \text{sum of all those treats which are } j^{th} \text{ associate of treatment } t_\alpha \text{ (for the } i^{th} \text{ associate group)} \]

\[ = S_j S_i (t_\alpha) \] \hspace{1cm} (2)
Hence from (1) and (2), \[ S_j S_i (t_\alpha) = \sum_{u=0}^{m} p_{ji}^u S_u (t_\alpha) \]

Intra block Analysis of two associate PBIBD: The reduced normal equation for the estimates of treatment effects are \[ Q = C \hat{t} \]

Where \[ C = R - NK^{-1} N' \]

Hence for 2-associate class PBIBD we get –

\[ = r I_v - k^{-1} NN' = r I_v - k^{-1} \sum_{i=0}^{m=2} \lambda_i B_i = r I_v - k^{-1} [ \lambda_0 B_0 + \sum_{i=1}^{2} \lambda_i B_i ] \]

\[ C = r I_v - k^{-1} [ r I_v + \sum_{i=1}^{2} \lambda_i B_i ] = r (1 - k^{-1}) I_v - k^{-1} \sum_{i=1}^{2} \lambda_i B_i \]

\[ Q = C \hat{t} = [ r (1 - k^{-1}) I_v - k^{-1} \sum_{i=1}^{2} \lambda_i B_i ] \hat{t} \]

\[ = r (1 - k^{-1}) \hat{t} - k^{-1} \sum_{i=1}^{2} \lambda_i B_i \hat{t} = r (1 - k^{-1}) \hat{t} - k^{-1} \sum_{i=1}^{2} \lambda_i S_i (\hat{t}) \]

\[ \therefore Q_s = r (1 - k^{-1}) \hat{t}_s - k^{-1} \sum_{i=1}^{2} \lambda_i S_i (\hat{t}_s) \]
for some treatment say, s
\[ kQ_s = r(k-1)\hat{t}_s - \sum_{i=1}^{2} \lambda_i S_i(\hat{t}_s) \]

\[ = r(k-1)\hat{t}_s - \lambda_1 S_1(\hat{t}_s) - \lambda_2 S_2(\hat{t}_s) \quad (1) \]

we know that, for testing \( \mu_0 : t = 0, E_{1v}t = 0 \)

\[ E_{1v}t = 0 \quad \Rightarrow \quad S_0(t_s) + S_1(t_s) + S_2(t_s) = 0 \]

\[ \therefore S_2(t_s) = -t_s - S_1(t_s) \]

Substituting \( S_2(t_s) \) in (1) we get,

\[ kQ_s = r(k-1)\hat{t}_s - \lambda_1 S_1(\hat{t}_s) - \lambda_2 [-\hat{t}_s - S_1(\hat{t}_s)] \]

\[ = [r(k-1) + \lambda_2]\hat{t}_s + (\lambda_2 - \lambda_1)S_1(\hat{t}_s) \]

\[ = A\hat{t}_s + BS_1(\hat{t}_s) \quad (2) \]

where, \( A = r(k-1) + \lambda_2 \) and \( B = \lambda_2 - \lambda_1 \),

If we put value of \( S_i(t_s) = -t_s - S_2(t_s) \) in (1) then

\[ kQ_s = [r(k-1) + \lambda_1]t_s + (\lambda_1 - \lambda_2)S_2(t_s) \]

\[ = A\hat{t}_s + BS_2(\hat{t}_s) \]
where, $A = r (k - 1) + \lambda_1$ and $B = \lambda_1 - \lambda_2$, now summing over (2) for the treatments which are first associate of treatment $\hat{t}_s$ we get

$$S_1 (kQ_s) = AS_1 (\hat{t}_s) + BS_1 (S_1 (\hat{t}_s))$$  \hspace{1cm} (3)

$$\therefore S_1 S_1 (\hat{t}_s) = \sum_{u=0}^{2} p_{11}^u S_u (\hat{t}_s)$$

$$= p_{11}^0 S_0 (\hat{t}_s) + p_{11}^1 S_1 (\hat{t}_s) + p_{11}^2 S_2 (\hat{t}_s)$$

$$= n_1 \hat{t}_s + p_{11}^1 S_1 (\hat{t}_s) + p_{11}^2 [-\hat{t}_s - S_1 (\hat{t}_s)]$$

$$= (n_1 - p_{11}^2) \hat{t}_s + (p_{11}^1 - p_{11}^2) S_1 (\hat{t}_s)$$

$$= p_{12}^2 \hat{t}_s + (p_{11}^1 - p_{11}^2) S_1 (\hat{t}_s) \therefore p_{12}^2 = n_1 - p_{11}^2$$

hence (3) reduce to

$$S_1 (kQ_s) = AS_1 (\hat{t}_s) + B[p_{12}^2 \hat{t}_s + (p_{11}^1 - p_{11}^2) S_1 (\hat{t}_s)]$$
\[ Bp_{12}^2 \hat{t}_s + \left[ A + B(p_{11}^1 - p_{11}^2) \right] S_1(\hat{t}_s) \]

where \( C = Bp_{12}^2 \) and \( D = \left[ A + B(p_{11}^1 - p_{11}^2) \right] \)

\[ DkQ_s = AD \hat{t}_s + BDS_1(\hat{t}_s) \]

\[ BS_1(kQ_s) = BC \hat{t}_s + BDS_1(\hat{t}_s) \]

\[ DkQ_s - BS_1(kQ_s) = (AD - BC) \hat{t}_s \]

\[ = X \hat{t}_s \]

Where \( X = AD - BC \)

\[ \hat{t}_s = \frac{1}{X} [DkQ_s - BS_1(kQ_s)] \]

Adjusted Treatment S.S. \[ \hat{t}' Q = \sum_{s=1}^{V} \hat{t}_s Q_s \]

\[ = \sum_{s=1}^{V} \frac{1}{X} [DkQ_s - BS_1(kQ_s)] Q_s \]
\[
= \frac{1}{X} \left[ \sum_{s=1}^{v} (DkQ_s^2 - BS_1(kQ_s))Q_s \right]
\]

\[
= \frac{1}{X} \left[ \sum_{s=1}^{v} \left( \frac{D}{k} (kQ_s)^2 - \frac{B}{k} S_1(kQ_s)(kQ_s) \right) \right]
\]

\[
= \frac{D}{kX} \sum_{s=1}^{v} (kQ_s)^2 - \frac{B}{kX} \sum_{s=1}^{v} (kQ_s)(S_1(kQ_s))
\]

\[
= \frac{1}{kX} \left[ D \sum_{s=1}^{v} (kQ_s)^2 - B \sum_{s=1}^{v} (kQ_s)S_1(kQ_s) \right] \quad (5)
\]

The other quantities are to be obtained as usual.

ANOVA TABLE

<table>
<thead>
<tr>
<th>source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.S.</th>
<th>MSS</th>
<th>S.S.</th>
<th>d.f.</th>
<th>source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block (unadj.)</td>
<td>b-1</td>
<td>1 ( \left( \sum B_j^2 \right) - C.F. )</td>
<td>SB/b-1</td>
<td>+</td>
<td>b-1</td>
<td>Block(adj.)</td>
<td></td>
</tr>
<tr>
<td>Treat(adj)</td>
<td>v-1</td>
<td>(S)</td>
<td>(S)/v-1</td>
<td>1 ( \sum T_i^2 - C.F. )</td>
<td>v-1</td>
<td>treat(unadj)</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>((bk-v-b+1))SSE</td>
<td>SSE/df(E)</td>
<td>(\sum y_i^2 + CT)</td>
<td>Error</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>bk - 1</td>
<td>(\sum y_i^2 - CT)</td>
<td>bk-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
ANALYSIS OF TWO WAY DESIGN

In each of block design the treatment are selected randomly which called one way block design, i.e. we remove the heterogeneity of data in the one direction. Now one is interested to remove it in to two direction so we need blocking in two ways, i.e., treatments are selected randomly in row and then treatments are selected randomly in columns which we called row – column design or two way heterogeneity of design or simply two way design.

The model of two way design is given by:

$$y_{jk}^{(i)} = \mu + \alpha_j + \beta_k + \tau_i + e_{jk}^{(i)}$$

(1)

when $y_{jk}^{(i)}$ is yield due to $j^{th}$ row and $k^{th}$ column for $i^{th}$ treatment.

$\mu =$ general mean ; $\alpha_j =$ effect of $j^{th}$ row

$\beta_k =$ effect of $k^{th}$ column ; $\tau_i =$ effect of $i^{th}$ treatment

$e_{jk}^{(i)} =$ random error in $j^{th}$ row and $k^{th}$ column for $i^{th}$ treatment.
Let there are \( n \) blocks arranged in \( u \) row and \( u' \) column. \( l_{ij} \) is a number of times \( v \) treatments are in \( u \) rows similarly, \( m_{ik} \) denotes the number of times \( v \) treatments are randomly allocated in \( u' \) column. Each treatment is replicated \( r_i \) times and each block contains \( k \) treatments.

\[ i = 1,2,...v; \ j = 1,2,....u; \ k = 1,2,.....u' \]

\[
l_{ij} = \begin{bmatrix}
    l_{11} & l_{12} & \cdots & l_{1u} \\
    l_{21} & l_{22} & \cdots & l_{2u} \\
    \vdots & \vdots & \ddots & \vdots \\
    l_{v1} & l_{v2} & \cdots & l_{vu}
\end{bmatrix} \begin{bmatrix}
    r_1 \\
    r_2 \\
    \vdots \\
    r_v
\end{bmatrix}
\]

\[ \therefore u' = \sum_i l_{ij} \quad \text{and} \]

Total \( u' \quad u' \quad \cdots \quad u' \)
\[
m_{ik} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1u'} \\ m_{21} & m_{22} & \cdots & m_{2u'} \\ \vdots & \vdots & \vdots & \vdots \\ m_{v1} & m_{v2} & \cdots & m_{vu'} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_v \end{bmatrix}
\]

\[
u \quad u \quad \cdots \quad u
\]

Assumptions:

- The model is linear.
- The model is fixed effect.
- The model is additive.
- The model is homoscedastic.

\[
e^{(i)}_{jk} \sim N(0, \sigma^2) \quad \text{And} \quad E[y^{(i)}_{jk}] = \mu + \alpha_j + \beta_k + \tau_i, \quad \text{var}(y^{(i)}_{jk}) = \sigma^2
\]

Here \(\mu, \alpha_j, \beta_k, \tau_i\) are unknown parameters which are to be estimated.

Let \(R = (R_1, \ldots, R_u); \quad C = (C_1, \ldots, C_u')\)
\[ T = (T_1, \ldots, T_v)' \]

\( R_j \) is the \( j^{th} \) row, \( C_k \) is the \( k^{th} \) column, and \( T_i \) is the \( i^{th} \) treatment effect.

Now, \( E = \sum_i \sum_j \sum_k e_{ijk}^{(i)2} = \sum_i \sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)^2 \)  \( (2) \)

\( \mu, \alpha_j, \beta_k \) and \( \tau_i \) are to be estimated from (2) using least square estimation.

\[ \frac{dE}{d\mu} = 2 \sum_i \sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0 \]

\[ \therefore \sum_i \sum_j \sum_k y_{jk}^{(i)} = \sum_i \sum_j \sum_k \mu + \sum_{ijk} \alpha_j + \sum_{ijk} \beta_k + \sum_{ijk} \tau_i \]

\[ G = uu'\mu + u'\sum_j \alpha_j + u\sum_k \beta_k + \sum_i uu'\tau_i \]

\[ = uu'\mu + u'\sum_j \alpha_j + u\sum_k \beta_k + \sum_i r_i \tau_i \]  \( (3) \)

\[ (\therefore \ uu' = \tau r_i = n) \]
\[
\frac{dE}{d\alpha_j} = 2 \sum_i \sum_k (y_{ik}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0
\]

\Rightarrow \sum_k y_{jk}^{(i)} = \sum_k \mu + \sum_k \alpha_j + \sum_k \beta_k + \sum \tau_i

\Rightarrow R_j = u'\mu + \mu'\alpha_j + \sum_k \beta_k + u'\tau_i

\Rightarrow R_j = u'\mu + u'\alpha_j + \sum k \beta_k + \sum l_{ij} \tau_i \quad (4)

\[
\frac{dE}{d\beta_k} = 2 \sum_j (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0
\]

\Rightarrow \sum_j y_{jk}^{(i)} = \sum_j \mu + \sum_j \alpha_j + \sum_j \beta_k + \sum \tau_i

\]

C_k = u\mu + \sum_j \alpha_j + u\beta_k + u\tau_i

= u\mu + \sum_j \alpha_j + u\beta_k + \sum m_{ik} \tau_i \quad (5)
\[
\frac{dE}{d\tau_i} = 2 \sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0
\]

\[
\Rightarrow \sum_j \sum_k y_{jk}^{(i)} = \sum_j \sum_k \mu + \sum_j \alpha_j + \sum_k \beta_k + \sum \tau_i
\]

\[
\Rightarrow T_i = uu'\mu + u'\sum_j \alpha_j + u \sum_k \beta_k + uu'\tau_i
\]

\[
= \sum_i r_i \mu + \sum_i l_{ij} \sum_j \alpha_j + \sum_i m_{ik} \sum_k \beta_k + \sum_i r_i \tau_i
\]

\[
= r_i \mu + \sum l_{ij} \alpha_j + \sum m_{ik} \beta_k + r_i \tau_i \quad (6)
\]

Now, we have 1+u+u'+v normal equations. This normal equations are dependent because if we sum over all \( j \) of (4), sum over all \( k \) for (5) and sum over all \( v \) for (6), we reach at (3). Hence the unknown parameters \( \mu, \alpha_j, \beta_k, \tau_i \) can not be estimated.
To estimate this parameters one has to put some restrictions. Let as convert this normal equation (3), (4), (5), (6) in the form of matrix.

\[
\begin{bmatrix}
G \\
R \\
C \\
T
\end{bmatrix} = \begin{bmatrix}
uu' & u'E_{1u} & uE_{1u} & r' \\
u'E_{u1} & u'I_u & Euu' & L' \\
u'E_{u1}' & Eu'u & ulu' & M' \\
r & L & M & \text{diag.}(r_1...r_v)
\end{bmatrix} \begin{bmatrix}
\mu \\
\alpha \\
\beta \\
\tau
\end{bmatrix}
\]  

The normal equation (7) can be expressed in the form \( A'y = A'A\hat{\theta} \)

We have \( \text{Var}(y) = \sigma^2 \text{In.} \)

\( \text{Var} (A'y) = A'\text{Var}(y)A = A'\sigma^2 \text{ I}_nA = \sigma^2 A'A \)
\[ \begin{bmatrix} G \\ R \\ C \\ T \end{bmatrix} \begin{bmatrix} uu' & u'E_{1u} & uE_{1u}' & r' \\ u'E_{u1} & u'I_u & E_{uu'} & L' \\ uE_{u1}' & Eu'u & uIu' & M' \\ r & L & M & \text{diag.}(r_1 \ldots r_v) \end{bmatrix} \]

To test the parameters \( H_0 : t = 0 \)

We assume that \( E_{1u} \alpha = 0 \), \( E_{1u'} \beta = 0 \), \( E_{1v} \gamma = 0 \)

From (7) we get,

\[ G = uu' \hat{\mu} + u'E_{1u} \hat{\alpha} + uE_{1u'} \hat{\beta} + r' \hat{t} \]

\[ = uu' \hat{\mu} + r' \hat{t} \]

\[ \therefore \hat{\mu} = \frac{1}{uu'} [G - r' \hat{t}] \quad (8) \]

\[ R = u'E_{u1} \hat{\mu} + u'I_u \hat{\alpha} + E_{uu'} \hat{\beta} + L' \hat{t} \]

\[ = \hat{\mu} u'E_{u1} + u' \hat{\alpha} + L' \hat{t} \]

\[ \therefore \hat{\alpha} = \frac{1}{u'} [R - \hat{\mu} uu'E_{u1} - L' \hat{t}] \quad (10) \]
\[
C = uE_{u'1} \hat{\mu} + E_{u'u} \hat{\alpha} + uI_{u'} \hat{\beta} + M' \hat{t}
\]

\[
= \hat{\mu}uE_{u'1} + u \hat{\beta} + M' \hat{t} \quad (11)
\]

\[
\hat{\beta} = \frac{1}{u} [C - \hat{\mu}uE_{u'1} - M' \hat{t}]
\]

\[
T = r\hat{\mu} + L\hat{\alpha} + M\hat{\beta} + \text{diag.}(r_1, \ldots, r_v) \hat{t}.
\]

\[
\therefore T = r\hat{\mu} + \frac{LR}{u'} - \hat{\mu}L_{u'1} - \frac{LL'\hat{t}}{u'} + \frac{MC}{u} - \hat{\mu}M_{u'1} - \frac{MM'\hat{t}}{u} + \text{diag.}(r_1, \ldots, r_v) \hat{t}
\]

\[
= \frac{LR}{u'} + \frac{MC}{u} - r\hat{\mu} + \text{diag.}(r_1, \ldots, r_v) \hat{t} - \frac{LL'\hat{t}}{u'} - \frac{MM'\hat{t}}{u}
\]

\[
= \frac{LR}{u'} + \frac{MC}{u} - \frac{rG}{uu'} + \frac{rr'\hat{t}}{uu'} + \text{diag.}(r_1, \ldots, r_v) \hat{t} - \frac{LL'\hat{t}}{u'} - \frac{MM'\hat{t}}{u}
\]

\[
Q = F\hat{t} \quad \text{, where} \quad Q = T - \frac{LR}{u'} - \frac{MC}{u} + \frac{rG}{uu'}
\]
and $F = \text{diag} \left( r_1, \ldots, r_v \right) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{rr'}{uu'}$

Evidently we get $E_{1v} Q = 0$

$E(Q) = Ft$

$\text{Var} (Q) = \sigma^2 F$

$E_{1v} F = 0'$ and $FE_{v1} = 0$

$\therefore \text{Rank} \left( F \right) = v - 1$

when $\text{Rank} \left( F \right) = v - 1$ all treatment contrasts are estimable
Test: $\mu_0: t = 0$

We have,

$$
\hat{\mu} = \frac{G}{uu'} - \frac{r' \hat{t}}{uu'} = \frac{1}{uu'} [G - r' \hat{t}]
$$

$$
\hat{\alpha} = \frac{R}{u'} - \hat{\mu} E_{ui} - \frac{L' \hat{t}}{u'} = \frac{1}{u'} [R - \hat{\mu} u'E_{u1} - L' \hat{t}]
$$

$$
\hat{\beta} = \frac{C}{u} - \hat{\mu} E_{u'1} - \frac{M' \hat{t}}{u} = \frac{1}{u} [C - \hat{\mu} u'E_{u'1} - M' \hat{t}]
$$

$$
SSR (\mu', \hat{\alpha}, \hat{\beta}, \hat{t}) = \hat{\theta}' A' y = \mu' G + \hat{\alpha} R + \hat{\beta} C + \hat{t}' T
$$

$$
= \hat{\mu} G + \frac{R' R}{u'} - \hat{\mu} E_{1u} R - \frac{\hat{t}' LR}{u'} + \frac{C' C}{u} - \hat{\mu} E_{1u'} C - \frac{\hat{t}' MC}{u} + \hat{t}' T
$$

$$
= \frac{R' R}{u'} - \hat{\mu} G + \hat{t}' T - \frac{\hat{t}' LR}{u'} + \frac{\hat{t}' MC}{u} + \frac{C' C}{u}
$$
\[
\frac{R' R}{u'} + \frac{C' C}{u} - \frac{G^2}{uu'} + \hat{t}'T + \frac{r' \hat{G}}{uu'} - \frac{\hat{t}'LR}{u'} - \frac{\hat{t}'MC}{u}
\]

\[
= \frac{R' R}{u'} + \frac{C' C}{u} - \frac{G^2}{uu'} + \hat{t}\left[T - \frac{LR}{u'} - \frac{MC}{u} + \frac{rG}{uu'}\right]
\]

\[
= \frac{R' R}{u'} + \frac{C' C}{u} - \frac{G^2}{uu'} + \hat{t}Q
\]

d.f. for \(SSR (\mu', \alpha, \beta, \hat{t}) = u + u' - 1 + \text{Rank} (F)\)

\[
= y'y - \text{SSR} (\mu', \alpha, \beta, \hat{t})
\]

\[
= y'y - \frac{R' R}{u'} - \frac{C' C}{u} + \frac{G^2}{uu'} + \hat{t}Q
\]

\[
= \left(y'y - \frac{G^2}{uu'}\right) - \left(\frac{R' R}{u'} - \frac{G^2}{uu'}\right) - \left(\frac{C' C}{u} - \frac{G^2}{uu'}\right) - \hat{t}Q
\]
d.f. for SSE are \( u u' - [u + u' - 1 + \text{Rank} (F)] \)
\[ = (u' - 1)(u - 1) - \text{Rank} (F). \]

now under \( :\mu_0 : t = 0 \), we get.

\[
SSR (\mu_0) = \theta^* A'y
\]
\[= SSR (\mu^*, \alpha^*, \beta^*)
\]
\[
= \frac{R' R}{u'} + \frac{C' C}{u} - \frac{G^2}{u u'}
\]

sum square due to \( :\mu_0 : t = 0 \) is
\[= SSR - SSR (\mu_0)
\]
\[= tQ.
\]

d.f. for SS due to \( :\mu_0 : t = 0 \) are \text{Rank}(F).

Hence the test for testing \( :\mu_0 : t = 0 \) is
\[
F = \frac{\hat{t}' Q/\text{Rank} (F)}{\text{SSE}/[(u - 1)(u' - 1) - \text{Rank} (F)]}
\]
ANOVA TABLE: (For testing $H_0 : \alpha = 0$)

<table>
<thead>
<tr>
<th>Sources</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.S.</th>
<th>$F_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows(unadj.)</td>
<td>$u-1$</td>
<td>$\frac{1}{u'} \sum R_j^2 - CT$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Column(unadj.)</td>
<td>$u'-1$</td>
<td>$\frac{1}{u'} \sum R_k^2 - CT$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treat(adj.)</td>
<td>Rank(F)</td>
<td>$\hat{t}'Q$</td>
<td>$\frac{\hat{t}'Q}{R(F)}$</td>
<td>$\frac{\hat{t}'Q}{R(F)}$</td>
</tr>
<tr>
<td>Error</td>
<td>$a$</td>
<td>$b$</td>
<td>$b/a(=S)$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>uu’-1</td>
<td>$\sum y^2 - CT$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

On similar lines one can obtain test for testing (i) $H_0 : \alpha = 0$
(ii) $H_0 : \beta = 0$
Particular Designs:
Latin square designs
Definition: A LSD is an arrangement of v treatments in \( v^2 \) plots arranged in \( v \) rows and \( v \) column that every treatment occurs exactly once in each rows and each column.
Remarks: (i) These designs require equal number of treatments and replication. In this case number of rows equal to number of column equal to number of treatments.
(ii) A LSD with \( v \) treatments is a LSD of order \( v \).
(iii) A LSD with symbols in first row and first column in natural order is called a regular LSD and a LSD with symbols in first row in natural order is called a semi-regular LSD.
Analysis: 
\[ H_0 : t = 0 \]
\[ u = v = u' \quad \text{and} \quad L = E_{vv} = M. \]
\[
Q = T - \frac{LR}{u'} - \frac{MC}{u} + \frac{rG}{vu'}
\]
\[
= T - \frac{LR}{u'} - \frac{MC}{u} + \frac{LE_{u_1}G}{vu'}
\]
\[
= T - \frac{E_{vv}R}{v} - \frac{E_{vv}C}{v} + \frac{E_{vv}E_{v_1}G}{v^2}
\]
\[
= T - \frac{GE_{v_1}}{v} - \frac{GE_{v_1}}{v} + \frac{GE_{v_1}}{v}
\]
\[
= T - \frac{GE_{v_1}}{v}
\]

In LSD we have \( r_1 = r_2 = \ldots = r_v = v \)

\[F = \text{diag} \left( r_1, \ldots, r_v \right) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{rr'}{uu'}\]
\[= vI_v - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{LE_{v_1}E_{1v}L'}{uu'}\]
\[
= vI_v - \frac{vE_{vv}}{v} - \frac{vE_{vv}}{v} + E_{vv}E_{vl}E_{lv}E_{vv} / vv \\
= vI_v - E_{vv} = v \left( I_v - \frac{1}{v} E_{vv} \right) \\
\therefore \text{Rank} (F) = v-1 \\
Q = F\hat{t} \\
= v \left( I_v - \frac{1}{v} E_{vv} \right)\hat{t} \\
= v\hat{t} - E_{vv}\hat{t} = v\hat{t} \\
\therefore \hat{t} = (1/v) \overline{Q} \\
\therefore \text{Adj. tr. S.S.} = \overline{t}'Q \\
= (1/v)Q'Q \\
= \frac{1}{v} \left[ T' - \frac{GE_{lv}}{v} \right]\left[ T - \frac{GE_{vl}}{v} \right] \\
= \frac{1}{v} \left[ T'T - \frac{GT'E_{vl}}{v} - \frac{GE_{lv}T}{v} + \frac{G^2v}{v^2} \right]
\[ \frac{1}{v} \left[ T'T - \frac{GG}{v} - \frac{GG}{v} + \frac{G^2v}{v^2} \right] \]

\[ = \frac{1}{v} \left( T'T - \frac{G^2}{v} \right) \]

\[ \text{Adj. tr. S.S.} = \frac{1}{v} \sum_{i=1}^{v} T_i^2 - \frac{G^2}{v^2} \]

**ANOVA TABLE:** (For testing: \( H_0 : t = 0 \))

<table>
<thead>
<tr>
<th>Sources</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.S.</th>
<th>( F_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows</td>
<td>( v-1 )</td>
<td>( \frac{1}{v} \sum R_i^2 - TC )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td>( v-1 )</td>
<td>( \frac{1}{v} \sum C_j^2 - TC )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treats</td>
<td>( v-1 )</td>
<td>( \frac{1}{v} \sum T_i^2 - TC )</td>
<td>MST</td>
<td>MST/MSE</td>
</tr>
<tr>
<td>Error</td>
<td>+</td>
<td>( \sum y^2 - CT )</td>
<td>MSE</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( v^2-1 )</td>
<td>( \sum y^2 - CT )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The test for testing: $\mu_0 : t = 0$ is $F_c = \frac{\hat{t}'Q/(v-1)}{MSE}$
on similar line one can obtain test for sig. For testing

(i) $H_0 : \alpha = 0$

(ii) $H_0 : \beta = 0$

Remarks: (i) We know that d.f. carried by SSE are $(v-1)(v-2)$.
Hence d.f. carried by SSE of a LS of order $v = 2$ are $(v-1)(v-2) = 0$.
Therefore for smaller number of treatments, d.f. for SSE of LSD are very few. Hence LSD`s are not suitable for smaller number of treatments.

(ii) The analysis of LSD become very much complicated when several plot yields are missing.

(iii) LSD`s less flexible.
CROSS OVER DESIGNS :-
We have seen that when number of treats are smaller, LSD`s are not suitable. In such situations Cross over designs are used. These designs are widely used in animal husbandry. Cross Over Designs resemble with LSD`s from analysis point of view, these are nothing but P replications of $v \times v$ LS`s. Thus variation from P replications is also eliminated from error s.s.

Let there be $v$ treatments arranged in $v$ rows and $vp$ columns such that there are $p$ replications of $v \times v$ LS.

Hence we get.

\[ u = v, \quad u' = vp; \quad l_{ij} = 1; \quad i = 1, 2, \ldots, v; \quad j = 1, 2, \ldots, v. \]

\[ :. L = E_{vv} \]

\[ m_{ik} = 1, \quad i = 1, 2, \ldots, v. \quad k = 1, 2, \ldots, vp \]

\[ :. M = E_{v(vp)} \]
\[ \theta = T - \frac{LR}{u'} - \frac{MC}{u} + \frac{RG}{uu'} \]

\[ = T - \frac{LR}{u'} - \frac{MC}{u} + \frac{LE_{ul}G}{uu'} \]

\[ = T - \frac{E_{vv}R}{vp} - \frac{E_{v(vp)}C}{v} + \frac{E_{vv}E_{ul}G}{v(vp)} \]

\[ = T - \frac{GE_{vl}}{vp} - \frac{GE_{vl}C}{v} + \frac{E_{vl}G}{vp} \]

\[ = T - \frac{GE_{vl}}{v} \]

\[ F = \text{diag. } (r_1, r_2, \ldots, r_v) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{rr'}{uu'} \]

\[ = vpI_v - \frac{E_{vv}E_{vv}}{vp} = \frac{E_{vvp}E_{vvp}}{v} + \frac{E_{vv}E_{vv}E_{vv}}{v(vp)} \]

\[ \therefore rr' = LE_{vl}E_{1v}L' \]

\[ = LE_{vv}L' \]
\[
= vpI_v - \frac{E_{vv}}{p} - pE_{vv} + \frac{E_{vv}}{p}
\]
\[
= vpI_v - pE_{vv} = vp[I_v - \frac{1}{v} E_{vv}]
\]
\[
\therefore \text{ Rank } (F) = v-1
\]
\[
Q = F\hat{t} = vp[I_v - \frac{1}{v} E_{vv}]\hat{t}
\]
\[
= vpI_v \hat{t} - pE_{vv} \hat{t}
\]
\[
= vp\hat{t}
\]
\[
\therefore \hat{t} = \frac{Q}{vp}
\]
\[
\text{Adj. tr. S.S.} = \hat{t}'Q = \frac{1}{v} pQ'Q
\]
\[
= \frac{1}{vp} [T' - GE_{v}/v] [T - GE_{v1}/v]
\]
\[
\frac{1}{vp} \left[ T' \bar{T} - \frac{GT'E_{vl}}{v} \right] - \frac{GE_{lv} \bar{T}}{v} + \frac{G^2E_{vl}E_{vl}}{v^2}
\]

\[
= \frac{1}{vp} \left[ T' \bar{T} - \frac{G^2}{v} \right]
\]

\[
= \frac{1}{vp} \left[ \sum_{i=1}^{v} T_i^2 - \frac{G^2}{v(vp)} \right]
\]
ANOVA TABLE: (For testing : $\mu_0 : t = 0$)

<table>
<thead>
<tr>
<th>Sources</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows</td>
<td>$v-1$</td>
<td>$\frac{1}{vp} \left( \sum_i R_i^2 \right) - (TC)$</td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td>$vp-1$</td>
<td>$\frac{1}{v} \sum_j C_j^2 - TC$</td>
<td></td>
</tr>
<tr>
<td>Treats</td>
<td>$v-1$</td>
<td>$\frac{1}{v} \sum T_i^2 - TC$</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>$(v-1)(vp-2)$</td>
<td>+ $\sum y^2 - CT$</td>
<td>$\text{MSE}$</td>
</tr>
<tr>
<td>Total</td>
<td>$v^2p-1$</td>
<td>$\sum y^2 - CT$</td>
<td></td>
</tr>
</tbody>
</table>

hence for testing $H_0 : t = 0$

$$F_c = \frac{\frac{1}{vp} \sum_{i=1}^{v} T_i^2 - CT}{(\theta - 1)}$$